Indefinite Causal Structures in Quantum Mechanics

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Abstract

We explore causal structures in physics and present recent results on the study of indefinite causal structures in quantum mechanics. These results include the causal inequality, nonseparable causal processes and results relating to computational speedup through the application of indefinite causal structures.

I. Introduction

Causality lies at the heart of physics. Indeed, the study of physical systems is fundamentally concerned with the characterisation of cause, effect and correlation. The topic of causality in quantum mechanics has recently attracted considerable attention due to a paper by Brukner *et al* in 2012 [14]. In it, the authors outlined the emergence of indefinite causal structures for a bipartite system in quantum mechanics. These investigations have partly been motivated by the long-lasting difficulties of formulating a consistent theory of quantum gravity. Discrepancies between the causal structures of quantum mechanics and general relativity (which we shall explore in the following sections) have led some to speculate that an even more fundamental theory might be required [**?**]. Apart from that, it has been speculated [8], and proven [7, **?**] that indefinite causal structures lead to computational speedup.

This report is intended as a brief survey of causality in physics and an introduction to the recent results on indefinite causal structures in quantum mechanics. Before we begin, let us first introduce some basic notation. A causal structure is defined based on the relationship between two (or more) events, here referred to as A and B. Where A causally precedes B we write $A \preceq B$, and similarly $B \preceq A$. We identify three types of causal structures that appear in physical theories; *fixed causal structures*, *dynamic causal structures*, and *indefinite causal structures*.

This report is structured as follows. In Section II we explore fixed causal structures and the role they play in Newtonian mechanics and quantum mechanics. Similarly, in Section III we study dynamic causal structures which arise in special and general relativity, and relativistic quantum field theory. With these pieces in place, we highlight some discrepancies between theories in Section IV and motivate why further studies of causality might yield interesting insights. We then introduce indefinite causal structures in Section V, by means of the causal inequality, followed by the process matrix formulation which leads us to non-separable causal processes. In addition, we make a short mention of computational speedups obtained through this formalism. Finally, Section VI provides a summary and an outline of some open questions.

II. Fixed Causal Structures

Given two events, A and B, a theory with a *fixed causal structure* does not provide any information about the causal order of A and B. In other words, the equations of motions tell us nothing about whether A causes B, or B causes A. Their causal relation emerges by assuming the existence of a fixed global background time. To gain further intuition, let us explore the fixed causal structure of Newtonian mechanics (NM) and quantum mechanics (QM).

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I. Newtonian Mechanics

Consider the simple case of two colliding billiard balls, a red one and a blue one. The red billiard ball has some initial momentum, which is then transferred to the blue ball through an elastic collision. The equation describing their interaction is given by Newton's 2nd Law,

$$F = ma. \tag{1}$$

The causal structure of the above example is not evident from 1 alone. It is not clear whether $F \preceq a$ or $a \preceq F$ - this relationship is only given by the initial assumption that the red billiard ball had initial momentum a, which then implies that $a \preceq F$.

Similarly, the equations of motion for a pendulum, or a Lagrangian used for minimisation problems do not provide any information about in which order events are causally related. Only by assuming some global background time t can we determine the causal order.

II. Quantum Mechanics

Given their commonly contrasting features, it might seem somewhat surprising that QM shares the fixed causal structure of NM. Consider two scientists, Alice and Bob, situated in two separate labs. Let us begin by assuming that the labs are spacelike separated and that Alice and Bob share the state ρ_{AB} between them. If Alice and Bob perform operations M_A and M_B respectively, we use the tensor product $\otimes$ to describe their action on the state, namely

$$\rho_{AB} \to M_A \otimes M_B \rho_{AB}. \tag{2}$$

We note that since non-local measurements are forbidden the assumption regarding the causal relationship between Alice and Bob necessitates the use of the tensor product.

Consider now a different situation where Alice wishes to send the state ρ to Bob. Their laboratories are timelike separated, and Alice can communicate ρ to Bob over an ideal noiseless channel. If Alice performs the operation M_A on ρ before sending it, and Bob subsequently performs M_B, we write

$$\begin{aligned} M_A\rho\Big|_{\text{Alice's Lab}} &\to M_A\rho\Big|_{\text{Bob's Lab}} \\ &\to M_A \cdot M_B\rho\Big|_{\text{Bob's Lab}}. \end{aligned} \tag{3}$$

Note the difference to the previous example - instead of the tensor product, we allow Alice's measurement to causally precede that of Bob by using matrix multiplication. This discrepancy of the two multiplication methods has been singled as a starting point out by Hardy who instead devised the so-called *causaloid product*, intended to unify the two structures and thus modify the fixed causal structure of QM [12].

In each scenario, we made an assumption about the causal structure *before* we were able to perform any calculations. The causal structure of QM is therefore fixed, which as we shall see later presents us with some difficulties in efforts to formulate a theory of quantum gravity.

III. Dynamic Causal Structures

Let us now investigate two theories with a *dynamic causal structure*, namely general relativity (GR) and relativistic quantum field theory (QFT). A theory with a dynamic causal structure requires no initial assumption about the ordering of events - the equations of motion will immediately causally structure the events for us.

I. Special and General Relativity

We start in flat spacetime, which allows us to use displacement vectors x^μ to denote events. We remind ourselves of Einstein's postulate, which states that the speed of light is the same in every inertial frame, which also places a limit on the speed of information transfer.

Let us now impose the restriction mathematically[1]. Doing so allows us to obtain the causal relationship between one event at the origin and one event at x^μ where $\mu = 0, 1, 2, 3$ by examining the quantity

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2, \tag{4}$$

where we have set $c = 1$. If we find that $(x^0)^2 > (x^1)^2 + (x^2)^2 + (x^3)^2$, the vector is timelike, whereas if

[1] A possibility which arises by treating time and space on an equal footing - something which is not possible in QM because time cannot be treated as an observable.

$(x^0)^2 \leq (x^1)^2 + (x^2)^2 + (x^3)^2$, the vector is spacelike or light-like for equality.

In GR, spacetime is not flat but curved, which complicates the attempt to recover the resulting causal structure. The curvature of spacetime is given by the *metric tensor*, written $g_{\mu\nu}$, which also provides us with a notion is distance. Instead of displacement vectors x^μ, we must now consider the curve of tangent vectors, $v^\mu = \frac{dx^\mu}{d\lambda}$. To learn whether two events can be causally related, the curve connecting them must be *causal* - that is, nowhere spacelike. Mathematically, we must make sure that the quantity $g(X, X)$ where X is a tangent vector is nowhere negative. The causal structure follows from the equations of GR without any need for an external assumption about a global background or pre-existing causal relationship between two events. [10]

II. Relativistic Quantum Field Theory

Does there exist a quantum equivalent of a dynamic causal structure? By going to infinite Hilbert spaces and expressing physical quantities not as quantum states ρ but as a field, $\phi(x)$, we are able to endow relativistic QFT with a dynamic causal structure. The causality requirement can succinctly be written down as [11],

$$[\phi(x), \phi(y)] = 0. \tag{5}$$

The commutator preserves causality in the following way. If two operators A and B do not commute, measuring A disturbs the measurement outcome of B. Thus, measuring the field[2] $\phi(x)$ at some point x disturbs the measurement outcomes of $\phi(y)$, which would allow for superluminal signalling.

We must therefore make sure that is everywhere zero for equal times. By going to second quantisation and writing the field in terms of annihilation and creation operators, a and $a^\dagger$, we find [16],

$$[\phi(x), \phi(y)] = \int \frac{d^3 p}{(2\pi)^3 E_p} \left(e^{-p(x-y)} - e^{ip(x-y)} \right), \tag{6}$$

where E_p is the energy, and x, y and p are position and momentum four-vectors. We then set $\vec{p} \to -\vec{p}$ and choose the negative energy solution from the relativistic energy relation,

$$E_p^2 = p^2 + m^2, \tag{7}$$

where we have again set $c = 1$. Thus, we obtain

$$\int \frac{d^3 p}{(2\pi)^3} \left(\frac{e^{-ip(x-y)}}{E_p} + \frac{e^{-ip(x-y)}}{-E_p} \right) = 0. \tag{8}$$

Remarkably, we conclude that relativistic QFT makes use of anti-particles to preserve causality. Outside the light cone-the potential superluminal propagation of a particle is cancelled out by the propagation of its anti-particle equivalent.

In summary, relativistic QFT has a dynamical causal structure where any non-local measurements are suppressed. [3] Unlike QM, the equation does not allow the conception of a non-local measurement.

IV. The Case for Causality

Now that all the pieces are in place, we can compare the causal structures of the various theories. We can summarise the contents of the previous section in the following table.

Theory	Causal structure
Newtonian Mechanics	Fixed
Quantum Mechanics	Fixed
Special and General Relativity	Dynamic
Relativistic Quantum Field Theory	Dynamic

We can immediately spot a major discrepancy [12] - the causal structure of QM is fixed, whereas the causal structure of GR is dynamic . Given that a theory of quantum gravity (QG) should eventually reduce to GR, starting from quantum theory alone will not work.

One might thus be inclined to choose relativistic QFT as a starting point, however numerous attempts have failed to produce experimentally verifiable predictions [**?**]. Our guiding question should thus be: Can we drop the assumption of a fixed causal structure in quantum mechanics? There exist several approaches which focus on causality with this question in mind [1, 13, 5, 12] but we shall here explore the existence of indefinite causal structures in QM.

[2] Technically, we never measure the field since we would then have to make a measurement over all space and time to measure the particles, which are represented by normal modes. What we do measure is the expectation values of the field.

[3] Note that the dynamic causal structure of relativistic QFT comes from the underlying classical field theory - there is nothing inherently quantum about the occurrence of antiparticles, they are perfectly valid objects in a classical field theory.

V. Indefinite Causal Structures

In contrast to a fixed or dynamic causal structure, a theory with an *indefinite causal structure* facilitates the existence of processes where the causal order is not known. In other words, an observer is not able to determine whether $A \preceq B$ or $B \preceq A$ by examining the process. As with many areas in QM, the conceptual understanding of this topic can be aided by formulating the operational objective as a game. We shall do so here by following the presentation in [14].

I. The Causal Inequality

Consider our two protagonists, Alice and Bob situated in separate labs. We assume that there exists a fixed causal structure between Alice and Bob, where Alice causally precedes Bob, $A \preceq B$. The goal of the game is for Alice and Bob to guess the other player's measurement result a, b. Before the game starts, a coin is tossed to produce either 0 or 1, which in turn determines whether Alice or Bob has to make the guess. They are then each given two states from which they obtain measurement outcome a or b respectively. Finally, either Alice or Bob produces a guess x or y.

Crucially, since Alice precedes Bob, Alice is allowed to signal to Bob[4]. To find out how often they can win the game, we maximise the probability of success, p_{succ}. Let us begin by analysing the situation where Bob has to make the guess (coin toss yields heads = 0). However, since Alice signalled to Bob and told him her measurement outcome, he can essentially perform perfectly. Thus, $p(y = a|0) = 1$. However, if Alice has to guess, she has no information whatsoever about Bob's measurement outcome, and her best strategy is therefore to guess randomly, giving $p(x = b|1) = \frac{1}{2}$. Putting everything together, we find

$$p_{succ} = \frac{1}{2}\left(p(y = a|0) + p(x = b|1)\right) \leq \frac{3}{4}. \tag{9}$$

This is the so-called *causal inequality*. Violating the causal inequality means disproving the assumption of a fixed causal structure. Naturally, our next step should be an attempt to find a process that does precisely that.

[4]This introduces a significant difference from other operational scenarios, such as the CHSH game, where signalling between the parties is strictly forbidden in order to obtain a loophole-free description of entanglement.

II. The Process Matrix

Our investigation will be greatly helped by constructing a new resource, the *process matrix* W, first introduced in [14]. W is motivated by the need to relate the input and output of quantum or classical systems and it can be seen as a generalisation of a quantum state. In other words, W describes the process which relates the input $A^0, B^0, C^0 \ldots$ of an unlimited number of players $A, B, C \ldots$ with the corresponding output $A^1, B^1, C^1 \ldots$.

The reader is directed to [14, 2] for a full mathematical derivation. In summary, the process matrix W is defined for the bipartite case by the relation between the measurement outcome $P\left(\mathcal{M}_i^A, \mathcal{M}_j^B\right)$ and the matrix $\mathcal{M}_i^{A_0A_1} \otimes \mathcal{M}_j^{B_0B_1}$, which through the Choi-Jamiolkowsky isomorphism represents the matrices describing the measurement process. We obtain,

$$P\left(\mathcal{M}_i^A, \mathcal{M}_j^B\right) = \mathrm{Tr}\left[W^{A_0A_1B_0B_1}\left(\mathcal{M}_i^{A_0A_1} \otimes \mathcal{M}_j^{B_0B_1}\right)\right]. \tag{10}$$

A so-called*separable causal process* can then be identified by being able to decompose W in the following manner:

$$W = qW^{A \not\preceq B} + (1-q)W^{B \not\preceq A}, \tag{11}$$

where $0 \leq q \leq 1$ and $A \not\preceq B$ reads 'A cannot signal to B'. These process matrices describe either a fixed causal structure (with q being either 0 or 1), or a superposition of causal processes. An example of such a superposition is the Quantum Switch (see next section).

However, we can also identify processes with *non-separable W*. That is, the process they describe *cannot* be written in a superposition of causal orders. Such a process is therefore referred to as *causally non-separable process*. It was shown in [14] that one such process violates the causal inequality by

$$p_{succ} = \frac{2+\sqrt{2}}{4} > \frac{3}{4}. \tag{12}$$

These causally non-separable processes have been further explored in [2], [4], and [9] where among other things a *causal witness*similar to an entanglement witness was found.

III. Computational Advantages

Aside from foundational incentives, it has been suggested that dropping the assumption of a fixed causal structure might yield certain computational advantages. Here we shall briefly outline some of them, beginning with the *Quantum Switch* [8]. The Quantum Switch can be understood as implementing a superposition of two fixed causal structures, represented by for instance two unitary gates U_A and U_B which are implemented either in the order $U_A U_B$ or $U_B U_A$. To decide whether $U_A U_B$ or $U_B U_A$ is implemented, we use a control qubit either prepared as $|0\rangle$ or $|1\rangle$ together with the state $|\psi\rangle$. The full action of the Quantum Switch is

$$V(U_A, U_B) = |0\rangle \langle 0| \otimes U_A U_B + |1\rangle \langle 1| \otimes U_B U_A. \quad (13)$$

The Quantum Switch can be used to probe in polynomial time $\mathcal{O}(n)$ whether the gates commute or anticommute. For any regular investigation, one would have to us one of the gates at least twice before the relationship is clear, bringing the task to $\mathcal{O}(n^2)$ for a fixed causal structure. The Quantum Switch was also recently experimentally implemented by Procopio *et al* [15]. It should be pointed out, however, that the Quantum Switch is a causally separable process, and therefore does not violate any causal inequality. [2]

VI. Conclusions

In this report, we have explored the underlying causal structure of some of the major physical theories, as well as given an introduction to indefinite causal structures in QM. We found that there exists non-separable causal processes that violate the causal inequality, and that implementations of indefinite causal structures can lead to computational advantages.

Many open questions currently surround the field. Firstly, many similarities between non-separable causal processes and entangled states have been found, and it remains to be seen whether more analogies can be found. Furthermore, Bell's inequality arises through spacelike separation between Alice and Bob, while the causal inequality relies on them being timelike separated. Recently, spatial and temporal correlations have been shown to be related by an isomorphism [6], which begs the question whether the causal inequality and Bell's inequality are fundamentally and intrinsically related.

There has also been some setbacks to the approach. It has been shown by Wofl *et al* that indefinite causal structures are not strictly a quantum phenomenon[3]. This, and the fact that the Quantum Switch and similar processes do not violate a causal inequality, it has been speculated as to whether a causally non-separable process can even be described as physical [2].

Despite these remaining difficulties, the study of indefinite causal structures has so far already yielded some fascinating insights into the foundations of quantum theory. Over the next few years, the validity and potential of the approach should become clear, which could potentially open up for many exciting possibilities.

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